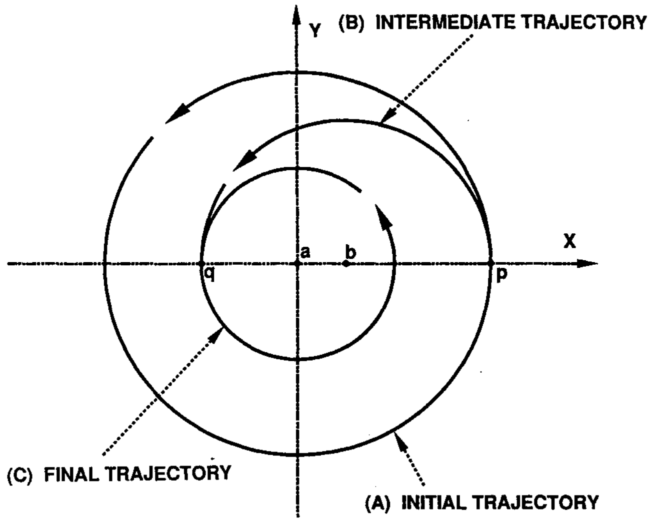


Fig. 2 Angular momentum history due to wheel pivoting during one nutation damping cycle.

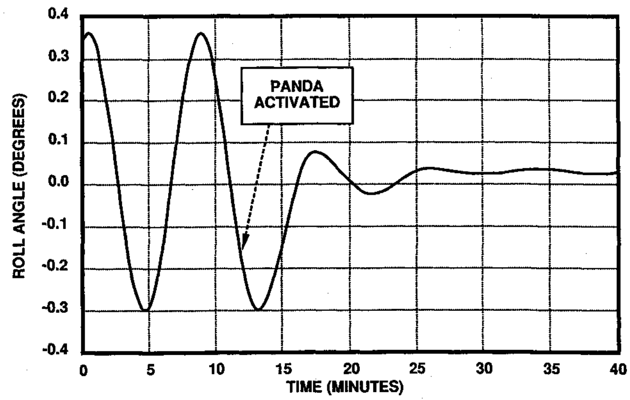


Fig. 3 Roll angle vs time.

path, $A$, centered on $a$. The motion along $A$ will be counterclockwise if the spacecraft is dynamically stable. At point $p$, where the roll rate crosses from negative to positive, the PANDA pivots the momentum wheel about $+Y$. This causes the equilibrium point to move from $a$ to $b$, and the momentum vector will now follow the circular trajectory $B$, which is centered at $b$. Half a nutation period later, when the momentum vector reaches point $q$, the momentum wheel pivots back to its original position. The momentum vector again circulates around point $a$, but the new circular trajectory, $C$, has a smaller radius than the original trajectory, $A$.

If the time required to pivot the wheel through its commanded angle is a small fraction of the nutation period, then the nutation reduction during one damping cycle is approximately twice the angle between the momentum vector orientations represented by points $a$ and $b$ in Fig. 2. Denoting this angle by $\phi$, it can be shown that (for small angles)

$$\phi = \theta h/[h + (I_z - I_x)\omega]$$

where $\theta$ is the wheel pivot angle, $h$ the wheel's angular momentum, $I_z$ the pitch moment of inertia, $I_x$ the yaw moment of inertia, and $\omega$ the spacecraft's pitch rate. During one damping cycle, the nutation angle is reduced by $2\phi$. If all of the angular momentum is in the wheel (i.e., if $\omega = 0$), then $\phi = \theta$, and the nutation angle is reduced by twice the pivot angle.

## Performance

Figures 3 and 4 show typical flight performance data. Here, the PANDA input is a pseudo roll rate derived from the integrated roll gyro output. Figure 3 shows the integrated gyro output (roll angle) vs time. The initial nutation angle is approximately 0.3 deg. Because Fig. 3 plots roll angle, not roll rate, the peak roll rates occur near the zero crossings, and the roll rate is zero at the plotted peaks.

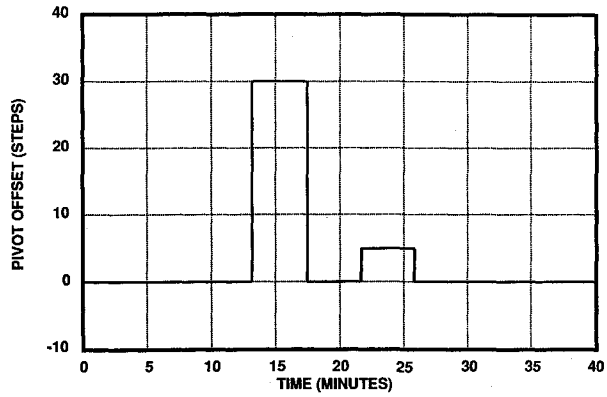


Fig. 4 Pivot offset vs time.

Figure 4 shows the pivot offset (in steps) vs time. On this spacecraft, the stepper motor pivots the wheel 0.00454 deg/step, with the steps occurring at 7.81 Hz. Because some nutation remained after the first damping cycle, a second cycle was required. The second damping cycle reduced the nutation below the 0.01 deg threshold.

The PANDA control logic was nominally designed to damp initial nutation of up to 1 deg or less within one nutation cycle. The flight data show, however, that two cycles were required. The second cycle was required because the PANDA was activated just after the peak roll rate, and, hence, the maximum roll rate detected by the PANDA logic was less than the actual sinusoidal amplitude. The control law commanded a wheel pivot angle proportional to this smaller rate, and only part of the nutation was damped in the first cycle. Had the PANDA been activated a minute earlier, the nutation would have been suppressed completely in the first cycle. Nevertheless, this performance confirms preflight simulations, which showed that no more than two damping cycles are required to suppress nutation that is initially 1 deg or less. These simulations also showed that if the initial nutation exceeds 1 deg the PANDA will remove approximately 1 deg/nutation cycle.

## References

[1]Beusch, J. U., Floyd, F. W., Much, C. H., Sferrino, V. J., and Smith, N. P., "Three Axis Attitude Control of a Synchronous Communications Satellite," AIAA Paper 70-456, April 1970.

[2]Much, C. H. and Smith, N. P., "Satellite Attitude Control with a Gimballed Reaction Wheel Digital Control System," *Automatica*, Vol. 8, 1972, pp. 9–21.

[3]Murakami, C. and Nakajima, A., "Active Nutation Control of Spacecraft Using Gimballed Momentum Wheel," International Astronautical Federation Paper 86-0244, Oct. 1986.

# Stability Analysis of Electro-Magnetoplasmadynamics

F. Pourki* and R. Shoureshi†

*Purdue University, West Lafayette, Indiana*

### Nomenclature

$C_v$ = constant volume specific heat

$C_p$ = constant pressure specific heat

Presented as Paper 87-0384 at the AIAA 25th Aerospace Sciences Meeting, Reno, NV, Jan. 12–15, 1987; received May 15, 1987; revision received March 22, 1988. 

*Graduate Research Assistant, School of Mechanical Engineering; currently, Assistant Professor, Department of Mechanical Engineering, Drexel University, Philadelphia, PA.

†Associate Professor, School of Mechanical Engineering. Member AIAA.

| | | |
|---|---|---|
| **E** | = | electric field vector |
| **H**,**h** | = | magnetic field vector and perturbation vector of magnetic field, respectively |
| $H_0$ | = | constant magnetic field or characteristic magnetic field |
| **J** | = | electric current density |
| $K$ | = | thermal conductivity of plasma |
| $L, \ell$ | = | characteristic length, length in the $x$ direction (for the flow) |
| $P,p'$ | = | pressure and perturbation in pressure, respectively |
| $R$ | = | gas constant |
| $T,T'$ | = | temperature and perturbation in temperature, respectively |
| $U$ | = | velocity |
| $u,v,w$ | = | velocity vector components in the $x$, $y$, and $z$ directions |
| $x,y,z$ | = | spatial coordinates |
| $x_j$ | = | spatial coordinates for $j=1,2,3$ |
| $\mu,\nu$ | = | viscosity and kinematic viscosity, respectively |
| $\mu_e$ | = | magnetic permeability |
| $\rho,\rho'$ | = | plasma density and perturbation in density, respectively |
| $\rho_e$ | = | excess electric charge/volume |
| $\sigma$ | = | electrical conductivity |
| $\nu_H$ | = | $1/\sigma\mu_e$ |

*Notation*

$T_h, T_w$ = time-dependent part of variable in subscript
$X_w, X_w$ = $x$-dependent part of variable in subscript

## Introduction

DYNAMICS of systems and processes that involve multi-energy modes and interactions among fluid, thermal, and electromagnetic fields generally can be expressed by partial differential equations. Such systems are called distributed parameter systems (DPS) by the fact that state variables are functions of both space and time. Electric propulsion systems such as magnetoplasmadynamic (MPD) engines are examples of such systems. In this paper, a Lyapunov method for stability analysis of such distributed parameter systems is presented.

The first attempts to apply Lyapunov's direct method for DPS were made by Massera[1] and Zubov.[2] An abstract theory for Lyapunov stability of an infinite-dimensional system was studied by Buis et al.[3] A DPS can be defined by the following equation:

$$\frac{\partial Z}{\partial t} = f\left(Z, \frac{\partial^n Z}{\partial X^n}, U, X, t\right) \tag{1}$$

where $Z(t,X)$ is an $n$-dimensional state vector, $X$ a spatial coordinate, $U$ the state input, and $t$ denotes time. A Lyapunov functional $V:\mathcal{R}^n \to \mathcal{R}$ is defined such that it satisfies the following:

1) $V(0)=0, \quad V(Z)>0$ for $Z \neq 0$ and $Z \epsilon \mathcal{R}^n$
2) $V \epsilon c^1(\mathcal{R}^n)$
3) $\dot{V} = \langle \text{grad} V, f \rangle \leq 0$

The following Lyapunov theorems are given.

*Theorem 1:* If in a neighborhood of the state origin there exists such a Lyapunov functional for system (1), satisfying 1–3, then the origin is a stable state for the system.

*Theorem 2:* If in a neighborhood of the origin there exists a Lyapunov functional $V$ for system (1), satisfying 1–3, also $V \to 0$ as $t \to \infty$, then the origin is an asymptotically stable state for the system.

## System Model

The governing equations for an MPD system consists of Maxwell's equations, Ohm's law, conservation of electric charge, equation of state (ideal gas law), and a set of mass, momentum, and energy equations.[4] Assume that the plasma is originally at rest with pressure $P_0$, temperature $T_0$, density $\rho_0$, and no electric field, but an external uniform magnetic field $H_0$ is applied to the system.

If the plasma is perturbed by a small disturbance, then the state of the system is a combination of equilibrium and perturbation states, hence the instantaneous pressure, temperature, density, electric and magnetic fields, and current density can be written as

$$P = P_0 + P'(x,t)$$
$$T = T_0 + T'(x,t)$$
$$\rho = \rho_0 + \rho'(x,t)$$
$$\mathbf{E} = \mathbf{i}\, E_x(x,t) + \mathbf{j}\, E_y(x,t) + \mathbf{k}\, E_z(x,t)$$
$$\mathbf{H} = \mathbf{i}\, [H_x + h_x(x,t)] + j\, [H_y + h_y(x,t)] + \mathbf{k}\, h_z(x,t)$$
$$\mathbf{J} = \mathbf{i}\, J_x(x,t) + \mathbf{j}\, J_y(x,t) + \mathbf{k}\, J_z(x,t)$$

In case of a neutral plasma, i.e., $\rho_e \simeq 0$, the number of ions and electrons per volume of plasma are nearly equal. By perturbation, one can derive a set of linearized governing dynamic equations of perturbed state.[4] If one considers the fact that $\partial h_x/\partial x = 0$ and $\partial h_x/\partial t = 0$, then it is possible to distinguish between two modes of wave propagation: transverse mode ($z$ direction) and longitudinal mode ($x$ direction).

### Transverse Mode

The governing equations for the transverse mode are as follows:

$$\frac{\partial Z}{\partial t} = AZ, \qquad Z = [h_z, w]^T, \qquad 0 \leq x \leq \ell, t \geq 0$$

$$A = \begin{bmatrix} \nu_H \frac{\partial^2}{\partial x^2} & H_x \frac{\partial}{\partial x} \\ \frac{V_x^2}{H_x} \frac{\partial}{\partial x} & \nu \frac{\partial^2}{\partial x^2} \end{bmatrix}$$

where $\nu_H = 1/\sigma\mu_e$ and $V_x = \sqrt{(\mu_e/\rho_0)}\, H_x$. The parameter $V_x$ is defined as the $x$ component of the Alfven wave speed.

### Longitudinal Mode

The state equations for this mode can be reduced to

$$\frac{\partial Z}{\partial t} = AZ, \qquad Z = [h_y, v, u, \rho'', T'']^T, \qquad 0 \leq x \leq \ell$$

$$A = \begin{bmatrix} \nu_H \frac{\partial^2}{\partial x^2} & H_x \frac{\partial}{\partial x} & -H_y \frac{\partial}{\partial x} & 0 & 0 \\ \frac{Vx^2}{H_x} \frac{\partial}{\partial x} & \nu \frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ -\frac{Vy^2}{Hy} \frac{\partial}{\partial x} & 0 & 4/3\nu \frac{\partial^2}{\partial x^2} & -RT_0 \frac{\partial}{\partial x} & -RT_0 \frac{\partial}{\partial x} \\ 0 & 0 & -\frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & -\frac{R}{C_v} \frac{\partial}{\partial x} & 0 & \frac{K}{\rho_0 C_v} \frac{\partial^2}{\partial x^2} \end{bmatrix}$$

where $\rho'' = \rho'/\rho_0$, $T'' = T'/T_0$, and $V_y = \sqrt{(\mu_e/\rho_0)}\, H_y$. The parameter $V_y$ is defined as the $y$ component of the Alfven wave speed.

## Lyapunov Functional and Stability Analysis

In this section, the Lyapunov functional approach is applied to each mode of the plasmadynamics. The stability results are derived and discussed.

1) Assuming that the boundary conditions are

$$Z(0,t)=0, \qquad Z(\ell,t)=0, \qquad Z(x,0)=Z_0 \in (L^2[0,\ell],E^2)$$

the operator $A$ is an infinitesimal of a semigroup. The domain of $A$ is dense in a Hilbert space, and the state vector function $Z(x,t) \in (L^2[0,\ell], E^2)$. A Lyapunov functional $V[Z(x,t)]$ can be constructed such that $V(Z) > 0$ and $dV/dt < 0$. For the problem, the functional $V$ is chosen to be the inner product of $Z$ (i.e., Hilbert norm of $Z$). As a result of the invariance of the stability property under equivalence norms, one can construct $V$ as a norm in another Hilbert space, $H_1$, such that $V$ is the equivalent norm of the original Hilbert space, $H_0$.

$$V=\langle Z,Z\rangle_1=\langle Z,PZ\rangle_0, \qquad P=\mathrm{diag}(\alpha_1,\alpha_2), \quad \alpha_1,\alpha_2>0$$

$$V=\alpha_1\|h_z\|^2+\alpha_2\|w\|^2$$

Hence,

$$\dot V=\frac{d}{dt}\langle Z,Z\rangle_1=2\langle \dot Z,Z\rangle_1=2\langle AZ,Z\rangle_1$$

Since $P$ is symmetric,

$$\langle x,Py\rangle=\langle Px,y\rangle, \qquad \dot V=2\langle AZ,PZ\rangle_0=2\langle PAZ,Z\rangle_0$$

$$\dot V=2\int_0^\ell\left\{h_z\left(\alpha_1\nu_H\frac{\partial^2 h_z}{\partial x^2}+\alpha_1 H_x\frac{\partial w}{\partial x}\right)+w\left(\frac{\alpha_2 V_x^2}{H_x}\frac{\partial hz}{\partial x}+\alpha_2\nu\frac{\partial^2 w}{\partial x^2}\right)\right\}dx$$

If $\alpha_1$ and $\alpha_2$ are selected such that $\alpha_1/\alpha_2=V_x^2/H_x^2=\text{const}$, then

$$\dot V=2\left[\alpha_1\nu_H\frac{\partial h_z}{\partial x}+\alpha_2\nu w\frac{\partial w}{\partial x}\right]_0^\ell-2\int_0^\ell\left[\alpha_1\nu_H\left(\frac{\partial h_z}{\partial x}\right)^2+\alpha_2\nu\left(\frac{\partial w}{\partial x}\right)^2\right]dx+2\,\alpha_1 H_x[h_z w]_0^\ell$$

Since $h_z(0,t)=h_z(\ell,t)=w(0,t)=w(\ell,t)=0$, then

$$\dot V=-2\int_0^\ell\left[\alpha_1\nu_H\left(\frac{\partial h_z}{\partial x}\right)^2+\alpha_2\nu\left(\frac{\partial w}{\partial x}\right)^2\right]dx<0$$

In a general case when $H_x$ is a function of $x$, following a tedious computation, one can find that $\dot V$ will consist of $(\partial h_z/\partial x)^2$ and $(\partial w/\partial x)^2$ as well as ($h_z{}^2$ and $(w)^2$). Then the estimate of $\dot V$ would require an estimate of $(\partial(\cdot)/\partial x)^2$ in terms of $(\cdot)^2$ for $h_z$ and $w$. Such an estimate can be given for the general parabolic problem as

$$\int_0^\ell\left(\frac{\partial f}{\partial x}\right)^2 dx \ge \pi^2\int_0^\ell f^2\,dx$$

If one choose $\alpha_2$ to be 1, then

$$\dot V=-2\left(\frac{V_x^2}{H_x^2}\nu_H\left\|\frac{\partial h_z}{\partial x}\right\|^2+\nu\left\|\frac{\partial w}{\partial x}\right\|^2\right)\le -2\pi^2\left[\frac{V_x^2}{H_x^2}\nu_H\|h_z\|^2+\nu\|w\|^2\right]$$

which indicates $\dot V$ is negative definite with respect to the norm $\langle z,z\rangle$. Therefore, stability of the transverse mode using the Lyapunov approach is satisfied.

2) For the longitudinal mode of wave propagation, the generic form of the evolution equation is considered, with

$$Z=[h_y,v,u,\rho'',T'']^T \in L^2(0,\ell),\ Z(0,t)=Z(\ell,t)=0.$$

An approach similar to the transverse mode is taken to construct the Lyapunov functional for the longitudinal mode:

$$V=\langle Z,\,PZ\rangle$$

where $P=\mathrm{diag}(\alpha_1,\alpha_2,\alpha_3,\alpha_5)$ and $\alpha_i>0$.

Introduction of this form for $V$ results in integral terms such as

$$\int_0^\ell Z_i\frac{\partial Z_j}{\partial x}\,dx$$

in $\dot V$. Such terms would make $\dot V$ indefinite unless solution forms are assumed for the state variables. In order to make the stability results independent of the solution forms, $\alpha_1$ is assumed to be unity with the following result:

$$\alpha_1\triangleq 1, \qquad \alpha_2=\frac{H_x^2}{V_x^2}, \qquad \alpha_3=\frac{H_y^2}{V_y^2}$$

$$\alpha_4=\frac{H_y^2}{V_y^2}RT, \qquad \alpha_5=\frac{H_y^2}{V_y^2}C_vT$$

Based on these values for $\alpha_i$ and similar algebraic techniques used in the transverse mode, the following can be derived:

$$V=\sum_{i=1}^5\alpha_i\|Z_i\|^2>0$$

and

$$\dot V=-2\left[\alpha_1\nu_H\left\|\frac{\partial h_y}{\partial x}\right\|^2+\alpha_2\nu\left\|\frac{\partial v}{\partial x}\right\|^2+\alpha_3 4/3\nu\left\|\frac{\partial u}{\partial x}\right\|^2+\alpha_5\frac{K}{\rho_0 C_v}\|T''\|^2\right]$$

also

$$\dot V\le -2\pi^2\left[\alpha_1\nu_H\|h_y\|^2+\alpha_2\,2\nu\|v\|^2+\alpha_3 4/3\nu\|u\|^2+\alpha_5\frac{K}{\rho_0 C_v}\|T''\|^2\right]$$

These results indicate that, for $Z_i$ and $\alpha_i \ne 0$, $V$ is positive and $\dot V$ is negative. Semidefinite with respect to the norm $V=\langle Z,PZ\rangle$. Therefore, the longitudinal mode is stable. Hence, stability of the MPD engine near its equilibrium is established without solving any of its dynamic equations.

Stability results for the transverse mode of the MPD engine also can be derived by application of the point spectrum approach. For comparison purposes, this derivation is presented.

By means of separation of variables, the semigroup property generated by $A$ can be constructed. The solution for

$$Z=\begin{bmatrix}T_hX_h\\ T_wX_w\end{bmatrix}$$

will exist if

$$\det\begin{bmatrix}\dfrac{\dot T_h}{T_h}-\nu_H\dfrac{X_h''}{X_h} & H_x\dfrac{X_w'}{X_w}\\[2ex] \dfrac{V_x^2}{H_x}\dfrac{X_h'}{X_h} & \dfrac{\dot T_w}{T_w}-\nu\dfrac{X_w''}{X_w}\end{bmatrix}=0$$

In order to have the $T$ and $X$ functions independent from $x$ and $t$, respectively, it is required that $\dot T/T$, $X'/X$, and $X''/X$ are constant. It is conclusive to represent $X$ in terms of a real periodic function. Hence, $X''/X=-\lambda 2/\ell^2$, $X'/X=i\lambda/\ell$, and

$\dot{T}/T = S$. From the boundary condition, it appears that $\lambda_n = n\pi$ for $n = \pm 1, \pm 2, \ldots$. Therefore, the characteristic equation can be reduced to

$$S^2 + S(\nu_H + \nu)\frac{\lambda_n^2}{\ell^2} + V_n^2\frac{\lambda_n^2}{\ell^2} + \nu\,\nu_H\frac{\lambda_n^4}{\ell^4} = 0$$

for the transverse mode with the roots

$$S_n\|_{1,2} = \tfrac{1}{2}\left[-(\nu_H + \nu)\frac{\lambda_n^2}{\ell^2} \pm\sqrt{(\nu_H - \nu)^2\frac{\lambda_n^4}{\ell^4} - 4V_x^2\frac{\lambda_n^2}{\ell^2}}\right]$$

This expression for $S_n$ indicates that $\sup[Re\,\sigma(A)] < 0$, which is the necessary and sufficient condition for the equilibrium solution of system equations to be exponentially stable, i.e., an equivalence to uniform asymptotic stability for the preceding linear system.

The point spectrum approach, applied to the longitudinal mode, results in the following spectrum of the set of state equations:

$$\left\{K\left(\frac{1}{\rho_0} + 4/3\frac{\nu}{p_0}S\right)\lambda_n^4 + \left[S^2\frac{K}{p_0} + 4/3\frac{\nu S^2}{T_0(\gamma-1)} + SC_p\right]\lambda_n^2 + \frac{S^3}{T_0(\gamma-1)}\right\}\left\{(\nu_H\lambda_n^2 + S)(\nu\lambda_n^2 + S) + V_x^2\lambda_n^2\right\} + \left\{\lambda_n^2V_y^2(S + \nu\lambda_n^2)\right\}\left\{\frac{S^2}{T_0(\gamma-1)} + \frac{S\,K\lambda_n^2}{p_0}\right\} = 0$$

where $p_0$ is the pressure related to $\rho_0$ and $T_0$, and $\gamma$ equal $C_p/C_v$.

This characteristic equation does not have a closed-form solution. Therefore, in general, the spectrum approach is very complex and requires a very cumbersome symbolic manipulation in order to give stability results except for very simplified or special cases. Also, in general, $\lambda$ can be both positive and negative numbers, which results in two sets of characteristic equations with positive and negative coefficients. The outlined Lyapunov approach however, would result in stability solutions for the system without any resort to the system parameters and the form of wave number.

## Conclusion

A stability analysis of a magnetoplasmadynamic system was presented. This technique is based on the Lyapunov theorem extended to cover distributed parameter systems. The results of the stability analysis were supported by those derived from a spectral analysis point of view. The procedure for construction of the Lyapunov functional and derivation of its derivative was presented. It was shown that although the spectral technique can be applied to stability analysis of a special class of systems, the presented method can be applied to any form of distributed parameter systems.

## Acknowledgment

This research has been supported under the U.S. Air Force Office of Scientific Research Grant 86-0278.

## References

[1]Massera, J. L., "Contributions to Stability Theory," *Annals of Mathematics,* Vol. 64, 1950, pp. 182–206.

[2]Zubov, V. I., *Methods of A. M. Lyapunov and their Applications,* Noordhoff, The Netherlands, 1964, Chap 5.

[3]Buis, G. R., Vogt, W. G., and Eisen, M. M., "Lyapunov Stability for Partial Differential Equations," NASA CR-1100, 1968.

[4]Pai, S. I., *Magneto-gas Dynamics and Plasma Dynamics,* Springer-Verlag, Vienna, 1962, Chap. 7.

# Calculation of Structural Dynamic Forces and Stresses Using Mode Acceleration

Paul Blelloch*

*SDRC, San Diego, California*

## Introduction

ONE challenge in structural dynamics is to accurately calculate displacement-related quantities such as element forces and stresses during a transient analysis while using a small number of modal degrees of freedom. A method for increasing the accuracy of the calculation is the use of mode acceleration rather than mode displacement data recovery.[1-3] This approach combines an exact static representation of the structure with a dynamic correction factor based on the modal accelerations. The standard mode acceleration formulation has often been interpreted to suggest that the reason for improved convergence (improved accuracy with a smaller number of modes) is that the dynamic correction factor is divided by the modal frequencies squared.[1,2] Here, we present an alternate formulation that indicates clearly that the only difference between mode acceleration and mode displacement data recovery is the addition of a static correction term. Reference 3 provides a correct interpretation of the method, though the alternate formulation presented here illustrates this more clearly. This alternate formulation also shows clearly that the use of the mode acceleration method is especially important when large input forces are present and, conversely, that mode acceleration and mode displacement data recovery are identical when input forces are not present (i.e., during free decay). We discuss some advantages in numerical implementation associated with the alternate formulation, and provide a simple example.

## Derivation and Discussion

The standard mode acceleration formulation for a structure with proportional damping is[1]

$$x = \Psi f - 2\Phi Z\Omega^{-1}\dot{q} - \Phi\Omega^{-2}\ddot{q} \tag{1}$$

where

| | |
|---|---|
| $x$ | = vector of displacement-related quantities (e.g., element forces or stresses) |
| $\Psi - \Psi_{ij}$ | = static response of $x_i$ due to a unit force $f_j$ |
| $f$ | = vector of input forces to the structure |
| $\Phi - \Phi_{ij}$ | = static response of $x_i$ due to unit deflection in the modal degree of freedom $q_j$ |
| $Z$ | = diagonal matrix of modal damping ratios, $Z = \mathrm{diag}\{\zeta_\theta\}$ |
| $\Omega$ | = diagonal matrix of modal frequencies, $\Omega = \mathrm{diag}\,\{\omega_i\}$ |
| $q$ | = vector of modal displacements |

The modal damping term is often neglected for lightly damped structures, but we will incorporate it in this formulation. Now consider the corresponding dynamic equations of motion in modal form:

$$\ddot{q} + 2Z\Omega\dot{q} + \Omega^2 q = \Gamma^T f \tag{2}$$

Received July 25, 1988; revision received Aug. 21, 1988. 

*Senior Project Engineer, Engineering Services.